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LETTER TO THE EDITOR

A reason for fusion rules to be even

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Abstract
We show that certain tensor product multiplicities in semisimple braided sovereign tensor categories must be even. The quantity governing this behaviour is the Frobenius–Schur indicator. The result applies in particular to the representation categories of large classes of groups, Lie algebras, Hopf algebras and vertex algebras.

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The tensor product of G-representations is a notion that exists for many algebraic structures G, such as groups, Hopf algebras and vertex algebras. In applications in mathematics and physics, often an important role is played by the space of invariants in a tensor product $V_{\mu_1} \otimes V_{\mu_2} \otimes \cdots \otimes V_{\mu_\ell}$ of irreducible G-modules V_{μ_i}.

In most cases there is in addition a notion of conjugate representation $V^\vee$. Irreducible modules which are self-conjugate, i.e. satisfy $V \cong V^\vee$, typically come in two classes—real (or orthogonal) modules and pseudo-real (or symplectic, or quaternionic) modules. They are distinguished by the so-called Frobenius–Schur indicator, which takes value $\nu = 1$ for real modules and $\nu = -1$ for pseudo-real modules.

For brevity, we refer to the dimension $\mathcal{N}_{\mu_1\mu_2\cdots\mu_\ell}$ of the space of invariants in the tensor product $V_{\mu_1} \otimes V_{\mu_2} \otimes \cdots \otimes V_{\mu_\ell}$ as a *fusion rule*, a terminology borrowed from two-dimensional conformal field theory. When dealing with representations of compact Lie groups, it is known that the fusion rule $\mathcal{N}_{\mu_1\mu_2\cdots\mu_\ell}$ is an even integer if all the irreducible modules V_{μ_i} are self-conjugate and the product $\prod_{i=1}^{\ell} \nu_{\mu_i}$ of their Frobenius–Schur indicators equals -1. In contrast, when the product of the Frobenius–Schur indicators is equal to +1, the value of the fusion rule is not restricted. To mention another example, recent studies of theories of unoriented strings and their underlying conformal field theories lead to the conjecture [1] that this relationship also holds for representations of (rational) vertex algebras.

In the present letter we show that this property of fusion rules can be formulated in a very general setting—(braided, semisimple) *sovereign tensor categories*—in which it also finds its natural proof. A sovereign tensor category $\mathcal{C}$ is a tensor category with a left and a right duality that coincide as functors. We take the tensor category $\mathcal{C}$ to be strict, so that the tensor product is strictly associative. (By the coherence theorems, this can be assumed without loss of generality.) For simplicity, we also assume that the morphism sets $\mathrm{Hom}(X, Y)$ of $\mathcal{C}$ are vector spaces over some field k.

A right duality assigns to every object X in the tensor category $\mathcal{C}$ another object $X^\vee$, called the right-dual object, and gives a family of morphisms

$$b_X \in \mathrm{Hom}(\mathbf{1}, X \otimes X^\vee) \qquad \text{and} \qquad d_X \in \mathrm{Hom}(X^\vee \otimes X, \mathbf{1}) \tag{1}$$

($\mathbf{1}$ denotes the tensor unit, satisfying $\mathbf{1} \otimes X = X = X \otimes \mathbf{1}$) such that the equalities

$$(\mathrm{id}_X \otimes d_X) \circ (b_X \otimes \mathrm{id}_X) = \mathrm{id}_X \qquad \text{and} \qquad (d_X \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_{X^\vee} \otimes b_X) = \mathrm{id}_{X^\vee} \tag{2}$$

hold. These data allow us to associate with every morphism $f \in \mathrm{Hom}(X, Y)$ its right-dual morphism

$$f^\vee := (d_Y \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_{Y^\vee} \otimes f \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_{Y^\vee} \otimes b_X) \in \mathrm{Hom}(Y^\vee, X^\vee). \tag{3}$$

Similarly, a left duality assigns a left-dual ${}^\vee X$ and gives morphisms

$$\tilde{b}_X \in \mathrm{Hom}(\mathbf{1}, {}^\vee X \otimes X), \qquad \tilde{d}_X \in \mathrm{Hom}(X \otimes {}^\vee X, \mathbf{1}) \tag{4}$$

that satisfy relations analogous to those in (2), and left-dual morphisms ${}^\vee f$ are defined analogously to (3).

A *sovereign tensor category* has both a left and a right duality which coincide both on all objects and on all morphisms: ${}^\vee X = X^\vee$ for all $X \in \mathcal{O}bj(\mathcal{C})$, and

$${}^\vee f = f^\vee \qquad \text{for all } f \in \mathrm{Hom}(X, Y) \tag{5}$$

and all $X, Y \in \mathcal{O}bj(\mathcal{C})$. Sovereignty is a strong property; in particular it allows one to define two notions $\mathrm{tr}_L(f)$ and $\mathrm{tr}_R(f)$ of a trace of an endomorphism f, both of which are cyclic. When applied to the identity morphism, these traces assign a left and a right (quantum) dimension $\dim_L(X) = \mathrm{tr}_L(\mathrm{id}_X)$ and $\dim_R(X) = \mathrm{tr}_R(\mathrm{id}_X)$, respectively, to each object X. (For our purposes, it is not necessary that the category is spherical, i.e. that the two traces coincide. For more information and references about dualities in tensor categories, see e.g. [2].) An immediate consequence of the definition of sovereignty is the simple, but useful, identity

$$\tilde{d}_Y \circ (g \otimes \mathrm{id}_{Y^\vee}) = d_Y \circ (\mathrm{id}_{Y^\vee} \otimes g) \tag{6}$$

that is valid for all objects Y of $\mathcal{C}$ and all $g \in \mathrm{Hom}(\mathbf{1}, Y)$.

We now restrict our attention to absolutely simple[3] objects X, i.e. objects for which the ring of endomorphisms coincides with the ground ring, $\mathrm{Hom}(X, X) \cong \mathrm{Hom}(\mathbf{1}, \mathbf{1}) = k$. In every sovereign tensor category there is the following notion of a Frobenius–Schur indicator of an absolutely simple self-dual object X: fix an isomorphism $\Phi \in \mathrm{Hom}(X, X^\vee)$ and consider

$$\mathcal{V}_X(\Phi) := (d_X \otimes \mathrm{id}_{X^\vee}) \circ (\mathrm{id}_{X^\vee} \otimes \Phi^{-1} \otimes \Phi) \circ (\mathrm{id}_{X^\vee} \otimes \tilde{b}_X) \in \mathrm{Hom}(X^\vee, X^\vee). \tag{7}$$

This morphism does not depend on Φ, and it must be a multiple of the identity:

$$\mathcal{V}_X(\Phi) = \nu_X \, \mathrm{id}_{X^\vee}. \tag{8}$$

The scalar ν_X is called the *Frobenius–Schur indicator* of X. If all left and right dimensions in the category are invertible, then ν_X can only take the values ± 1. In the definition of $\mathcal{V}_X$ one may exchange the role of left and right dualities, but by the sovereignty of $\mathcal{C}$ the two definitions are equivalent. The following identity, valid for all morphisms $\Phi \in \mathrm{Hom}(X, X^\vee)$, follows immediately from the definitions:

$$\tilde{d}_X \circ (\mathrm{id}_X \otimes \Phi) = \nu_X d_X \circ (\Phi \otimes \mathrm{id}_X). \tag{9}$$

[3] In many situations, e.g. when $\mathcal{C}$ is Abelian with morphism spaces that are finite-dimensional complex vector spaces, the notions of an absolutely simple and of a simple object are equivalent.

Having explained the setting, we can formulate our main result:

Theorem. Let $\mathcal{C}$ be a semisimple braided sovereign tensor category, $\mathbf{1} \in \mathcal{O}bj(\mathcal{C})$ the tensor unit, and $X_i \in \mathcal{O}bj(\mathcal{C})$ $(i \in \{1, 2, \dots, \ell\})$ self-dual absolutely simple objects with Frobenius–Schur indicators $\nu_i \in \{\pm 1\}$. Then the morphism space

$$\mathcal{H} := \mathrm{Hom}(\mathbf{1}, X_1 \otimes X_2 \otimes \cdots \otimes X_\ell) \tag{10}$$

can be endowed with a non-degenerate bilinear pairing.

This pairing is symmetric if the product $\nu := \prod_{i=1}^{\ell} \nu_i$ is +1, and antisymmetric if $\nu = -1$. As a consequence, for $\nu = -1$ the vector space $\mathcal{H}$ is even-dimensional.

Let us describe this bilinear form explicitly for the case of finite-dimensional self-conjugate simple representations $V_{\lambda_1}, V_{\lambda_2}, \dots, V_{\lambda_\ell}$ of a compact Lie group G on a complex vector space. It is well known (cf e.g. proposition II.6.4 in [3]) that in this case there is a family of non-degenerate, G-invariant bilinear forms

$$\omega_{\lambda_i} : V_{\lambda_i} \times V_{\lambda_i} \to \mathbb{C},$$

where ω_{λ_i} is symmetric for $\nu_{\lambda_i} = 1$ and skew-symmetric for $\nu_{\lambda_i} = -1$. It follows from Schur's lemma that these forms are unique up to a scalar. The product ω of these forms gives a non-degenerate, G-invariant bilinear form on

$$V_{\vec{\lambda}} := V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_\ell}$$

that is obviously symmetric for $\nu = 1$ and skew-symmetric for $\nu = -1$. Due to its invariance under G, it descends to a non-degenerate bilinear form on the G-invariants of $V_{\vec{\lambda}}$ with the same symmetry properties. This is precisely the bilinear form of the theorem.

As a matter of fact, we shall prove a slightly more general statement. To formulate it, it is convenient to define the Frobenius–Schur indicator of a non-self-dual absolutely simple object X to be zero, $\nu_X = 0$ for $X \not\cong X^\vee$.

Theorem. Let $\mathcal{C}$ be a semisimple braided sovereign tensor category, $\mathbf{1} \in \mathcal{O}bj(\mathcal{C})$ the tensor unit, and $X_i \in \mathcal{O}bj(\mathcal{C})$ $(i \in \{1, 2, \dots, \ell\})$ absolutely simple objects with Frobenius–Schur indicators $\nu_i \in \{0, \pm 1\}$. Suppose there exists a permutation $\pi \in S_\ell$ such that $X_{\pi(i)} \cong X_i^\vee$ for $i = 1, \dots, \ell$. Then the morphism space

$$\mathcal{H} := \mathrm{Hom}(\mathbf{1}, X_1 \otimes X_2 \otimes \cdots \otimes X_\ell) \tag{11}$$

can be endowed with a non-degenerate bilinear pairing.

This pairing is symmetric if the product $\nu := \prod_{i=1}^{\ell}(1 + \nu_i - \nu_i^2)$ is +1, and antisymmetric if $\nu = -1$. In particular, for $\nu = -1$ the vector space $\mathcal{H}$ is even-dimensional.

Proof. (1) The presence of the permutation π allows us to find another permutation $\sigma \in S_\ell$ of order two satisfying $X_{\sigma(i)} \cong X_i^\vee$ as well.

To see this, we decompose π in disjoint cycles $(i_1, \dots, i_n)$. On each such cycle we have $X_{i_k} \cong X_{i_{k+1}}^\vee \cong X_{i_{k+2}}$ (we cyclically identify $i_{n+1} = i_1$ and $i_{n+2} = i_2$). If the length n of a cycle is odd, then $X_{i_k} \cong X_{i_{k+1}}$ and hence all objects on the cycle are self-dual. On such a cycle we let σ act as the identity. If the order n is even, we take $\sigma(i_k) := i_{k+1}$ for even k and $\sigma(i_k) := i_{k-1}$ for odd k. Note that $\sigma(i) \neq i$ does not rule out the possibility that X_i is self-dual.

(2) Next we fix isomorphisms $f_i \in \mathrm{Hom}(X_i, X_{\sigma(i)}^\vee)$: if $\sigma(i) = i$, we pick an arbitrary isomorphism $f_i \in \mathrm{Hom}(X_i, X_i^\vee)$. For σ-orbits of length 2, we choose an arbitrary isomorphism $f_i \in \mathrm{Hom}(X_i, X_{\sigma(i)}^\vee)$ for one element i on the orbit, and then define $f_{\sigma(i)} \in \mathrm{Hom}(X_{\sigma(i)}, X_i^\vee)$ as

$$f_{\sigma(i)} := (\mathrm{id}_{X_i^\vee} \otimes d_{X_{\sigma(i)}}) \circ (\mathrm{id}_{X_i^\vee} \otimes f_i \otimes \mathrm{id}_{X_{\sigma(i)}}) \circ (\tilde{b}_{X_i} \otimes \mathrm{id}_{X_{\sigma(i)}}). \tag{12}$$

(Notice that $f_{\sigma(i)}$ is not the (left- and right-) dual morphism of f_i.) This definition does not depend on the choice of i on the orbit. Indeed, formula (12) implies that

$$f_i = (\mathrm{id}_{X^\vee_{\sigma(i)}} \otimes d_{X_i}) \circ (\mathrm{id}_{X^\vee_{\sigma(i)}} \otimes f_{\sigma(i)} \otimes \mathrm{id}_{X_i}) \circ (\tilde{b}_{X_{\sigma(i)}} \otimes \mathrm{id}_{X_i}), \tag{13}$$

which is shown by applying (6) to the morphism $g := (\mathrm{id}_{X_i^\vee} \otimes f_i) \circ \tilde{b}_{X_i} \in \mathrm{Hom}(\mathbf{1}, X_i^\vee \otimes X^\vee_{\sigma(i)})$. With this choice of the isomorphisms f_i, we have

$$d_{X_{\sigma(i)}} \circ (f_i \otimes \mathrm{id}_{X_{\sigma(i)}}) = p_i \tilde{d}_{X_i} \circ (\mathrm{id}_{X_i} \otimes f_{\sigma(i)}), \tag{14}$$

where $p_i = \nu_i$ if $\sigma(i) = i$ and $p_i = 1$ if $\sigma(i) \neq i$.

(3) We now construct a bilinear pairing $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to k \equiv \mathrm{Hom}(\mathbf{1}, \mathbf{1})$ out of the following two ingredients. First, the duality morphism

$$d_{X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(\ell)}} \in \mathrm{Hom}(X^\vee_{\sigma(1)} \otimes X^\vee_{\sigma(2)} \otimes \cdots \otimes X^\vee_{\sigma(\ell)} \otimes X_{\sigma(\ell)} \otimes X_{\sigma(\ell-1)} \otimes \cdots \otimes X_{\sigma(1)}, \mathbf{1}), \tag{15}$$

which may be obtained recursively by

$$d_{X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(\ell)}} = d_{X_{\sigma(1)}} \circ [\mathrm{id}_{X^\vee_{\sigma(1)}} \otimes d_{X_{\sigma(2)} \otimes X_{\sigma(3)} \otimes \cdots \otimes X_{\sigma(\ell)}} \otimes \mathrm{id}_{X_{\sigma(1)}}] \tag{16}$$

from the duality morphisms of the simple objects X_i. Second, any isomorphism

$$c_{X_1 \otimes X_2 \otimes \cdots \otimes X_\ell} \in \mathrm{Hom}(X_1 \otimes X_2 \otimes \cdots \otimes X_\ell, X_{\sigma(\ell)} \otimes X_{\sigma(\ell-1)} \otimes \cdots \otimes X_{\sigma(1)}) \tag{17}$$

that is a combination of braidings. (Any of the various possible combinations of braidings may be chosen; the argument does not depend on this choice.) Then the pairing $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is given by

$$\langle \phi, \phi' \rangle := d_{X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(\ell)}} \circ [f_1 \otimes f_2 \otimes \cdots \otimes f_\ell \otimes c_{X_1 \otimes X_2 \otimes \cdots \otimes X_\ell}] \circ [\phi \otimes \phi']. \tag{18}$$

(4) The pairing (18) is non-degenerate.

Since both $c_{X_1 \otimes X_2 \otimes \cdots \otimes X_\ell}$ and $f_1 \otimes f_2 \otimes \cdots \otimes f_\ell$ are isomorphisms, this follows from the fact that in every semisimple sovereign category for any object W the pairing

$$B : \begin{array}{l} \mathrm{Hom}(\mathbf{1}, W^\vee) \otimes \mathrm{Hom}(\mathbf{1}, W) \to k \\ \qquad\qquad \phi_1 \otimes \phi_2 \mapsto d_W \circ (\phi_1 \otimes \phi_2) = \phi_1^\vee \circ \phi_2 \end{array} \tag{19}$$

is non-degenerate. That is, as $\mathcal{C}$ is semisimple, the object W can be written as a direct sum $W \cong \bigoplus_i W_i$ of simple objects. Thus when $\phi_2 \in \mathrm{Hom}(\mathbf{1}, W)$ is non-vanishing, then at least one component corresponding to some $W_i \cong \mathbf{1}$ is non-vanishing, and there exists a ϕ_1 with a matching component such that $B(\phi_1, \phi_2)$ is non-zero.

(5) We finally analyse the symmetry properties of the pairing.

Using property (14) of the morphisms f_i as well as functoriality of the braiding, we obtain

$$\langle \phi, \phi' \rangle = \nu \tilde{d}_{X_{\sigma(\ell)} \otimes X_{\sigma(\ell-1)} \otimes \cdots \otimes X_{\sigma(1)}} \circ [c_{X_1 \otimes X_2 \otimes \cdots \otimes X_\ell} \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_\ell] \circ [\phi \otimes \phi'] \tag{20}$$

with $\nu = \prod_{i=1}^{\ell} p_i$. We wish to rewrite ν in terms of Frobenius–Schur indicators. For $\sigma(i) = i$ we have $p_i = \nu_i \in \{\pm 1\}$ and thus $p_i = 1 + \nu_i - \nu_i^2$. For $\sigma(i) \neq i$ we have $p_i = p_{\sigma(i)} = 1$, and in all three possible cases—that is, $X_i \cong X_i^\vee$ with $\nu = 1$ or with -1, or $X_i \not\cong X_i^\vee$—one has $(1 + \nu_i - \nu_i^2)(1 + \nu_{\sigma(i)} - \nu^2_{\sigma(i)}) = 1 = p_i p_{\sigma(i)}$. Altogether we obtain $\nu = \prod_{i=1}^{\ell} (1 + \nu_i - \nu_i^2)$. Applying the identity (6) to the expression (20), we thus find

$$\langle \phi, \phi' \rangle = \nu \langle \phi', \phi \rangle, \tag{21}$$

which proves the assertion about the (anti)symmetry of the pairing. □

Remark. While sovereignty of $\mathcal{C}$ is a crucial ingredient of the proof, the role of semisimplicity and of the braiding is limited.

(1) Instead of semisimplicity, a substantially weaker property is sufficient: a weak form of *dominance* [4], namely the existence of a family I of absolutely simple objects such that the identity morphism id_W of any object W can be decomposed in a finite sum $f = \sum_r g_r \circ h_r$ with $h_r \in \mathrm{Hom}(W, i)$ and $g_r \in \mathrm{Hom}(i, W)$ for some $i = i(r) \in I$.

(2) Concerning the braiding, all that is needed in the proof is the existence of an isomorphism $\tilde{c} \in \mathrm{Hom}(X_1 \otimes X_2 \otimes \cdots \otimes X_\ell, X_{\sigma(\ell)} \otimes X_{\sigma(\ell-1)} \otimes \cdots \otimes X_{\sigma(1)})$ such that

$$(f_{\sigma(\ell)} \otimes f_{\sigma(\ell-1)} \otimes \cdots \otimes f_{\sigma(1)}) \circ \tilde{c} = \tilde{c}^\vee \circ (f_1 \otimes f_2 \otimes \cdots \otimes f_\ell). \tag{22}$$

In a general sovereign tensor category without braiding, an isomorphism with this property need not exist. In fact, without any further structure there is even no reason that the two tensor products in question should be isomorphic at all. However, in every sovereign tensor category there is a class of tensor products to which the theorem applies immediately: those for which $X_j^\vee \cong X_{\ell-j+1}$ for all $j = 1, 2, \ldots, \ell$. In these cases, one may simply take $\tilde{c} = \mathrm{id}$ and $\sigma(j) = \ell - j + 1$.

Our theorem applies in particular to the category of finite-dimensional representations of a finite group or, more generally, of a finite-dimensional compact Lie group. In this case the statement is well known at least to experts. Apart from these classical applications, our result applies also to such modular tensor categories [4] in which the ground ring $\mathrm{Hom}(\mathbf{1}, \mathbf{1})$ is a field. Examples of modular tensor categories are the representation category of the quantum double of a finite group, or the (truncated) representation category of the deformed enveloping algebra of a simple Lie algebra (a quantum group) with the deformation parameter being a root of unity.

Many more examples of modular tensor categories are supplied by rational conformal field theories [5, 6]. In this context, the Frobenius–Schur indicator also shows up when one studies the conformal field theory on a Klein bottle (see [7, 8]). Correlators on the Klein bottle appear in the so-called orientifold projection (see e.g. [9]). As observed in [1], consistency of this projection requires certain coupling spaces to be even-dimensional. This observation amounts to the statement of our theorem for representations of rational vertex algebras, which provided our original motivation for studying this issue.

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